Structure and Localization of Solutions of Antennas Synthesis Problems with Flat Radiating Aperture According to the Prescribed Amplitude Directivity Pattern

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Abstract

The method of investigation and numerical solution of synthesis problems of antennas with flat radiating aperture according to the prescribed requirements to amplitude directivity pattern (DP) is proposed in this paper. Freedom of choice of the phase DP is used as an additional degree of freedom to improve the quality of approximation of the amplitude of the synthesized DP to the given one. It is shown that the non-uniqueness and branching of solutions, dependent on the size of two physical parameters that describe the size of antenna aperture, and the properties of given amplitude DP is characteristic features of this class of problems. The method of finding the branching lines of solutions of basic equations of synthesis, which is based on implicit function method, is proposed. This method allows to reduce a non-linear two-parameter spectral problem to solution of the corresponding Cauchy problem for differential equations of first order. On this basis the methodology of localization of existing solutions is developed. Effective computational algorithms for finding the optimal solutions of synthesis problems are constructed, the numerical examples are shown. Analysis of the effectiveness of different types of existing solutions for synthesis of several types of given amplitude DP is presented.

Keywords

Nonlinear Synthesis of Antennas; Flat Aperture; Nonuniqueness and Branching Of Solutions; Branching Lines; Nonlinear Two-Parameter Spectral Problem; Algorithms for Finding the Optimal Solutions

Introduction

As is known [1], in many practical applications at the design stage of antennas the requirements are only to the energy characteristics of antenna (amplitude DP or DP by the power). Here freedom of choice of the phase DP can be used to improve the quality of approximation of synthesized DP to the given one. The synthesis problems of antennas according to the prescribed amplitude DP or DP by power belong to the class of problems with incomplete input information. Nonuniqueness and branching (or bifurcation) of existing solutions is characteristic feature of this class of problems.

The problem of nonuniqueness and branching of existing solutions, determination of their general structure and main properties is investigated only partly for the synthesis problems of linear antennas and antenna arrays [2, 3]. It is shown that the quantity of existing solutions and their properties depend on the physical parameter c characterizing the length of antenna and solid angle, in which the prescribed amplitude DP is given, and properties of this DP. It is found that for arbitrary (bounded) nonnegative values of the parameter c there exist several types of solutions, belonging to the class of in-phase DP (called as primary). These solutions are optimal only at small values c. With the increase of the values c there exist such values c_i the points of branching in which the more effective complex solutions branch-off from the primary (real) solutions.

Note, the results obtained in [2, 3] may not be applicable to the synthesis problems of antenna with flat radiating aperture, since in these problems the solutions depend on two physical parameters and in contrast to the branching points (specific to the problem of synthesis of linear antennas and antenna arrays) there exist the branching lines of solutions. Moreover the problem of finding them is insufficiently studied nonlinear two-parameter spectral problem. The existence of connected

components of the spectrum, which in the case of real parameters are spectral lines, is an essential feature of nonlinear two-parameter problems.

In mathematical aspect in variational statements the synthesis problems of antenna with flat radiating aperture are reduced to study two-dimensional nonlinear integral equations of the Hammerstein type. Accordingly to the general branching theory of solutions corresponding linear homogeneous integral equation with a nonlinear occurrence of two spectral parameters in the kernel of operator to find the branching lines is obtained.

A numerical method for solving the nonlinear two-parameter spectral problems that allows to find the branching lines of solutions is proposed and justified in the work. Methods of the general branching theory of solutions allow to find the branching-off solutions. Jointly this gives opportunity to locate the existing solutions of the basic equations of synthesis and build effective calculation algorithms for finding and analysis of optimal solutions.

Formulation of problem. Basic equations of synthesis

According to [2] we shall formulate the synthesis problem of antennas with a flat radiating aperture $\overline{G}$ putting, that radiator is acoustic flat antenna placed in the plane XOY, and its DP is described by the formula [4]

$$f(\theta, \varphi) = AU \equiv \iint\limits_{\overline{G}} U(x, y) e^{ik(x \sin\theta\cos\varphi + y \sin\theta\sin\varphi)} dxdy . \quad (1)$$

Here $U(x, y)$ is the distribution of oscillating velocity on a

flat surface of antenna, $k = 2\pi/\lambda$ is a wave number, λ and is the wave length.

Introduce the generalized angular coordinates

$$s_1 = (\sin\theta\cos\varphi)/\sin\gamma_1 , \quad s_2 = (\sin\theta\sin\varphi)/\sin\gamma_2 \quad (2)$$

and the corresponding nondimensional parameters:

$$c_1 = ka_1 \sin\gamma_1, \qquad c_2 = ka_2 \sin\gamma_2 , \quad (3)$$

setting that antenna (aperture $\overline{G}$) (Fig. 1) is in some rectangle $\overline{G}_P = \{|x| \le a_1, |y| \le a_2\}$, the geometric center of which coincides with the beginning of coordinates and the sides are parallel to the axes OX, OY. Put that amplitude DP $F(\theta, \varphi)$ is given also in the domain $\overline{\Omega}$ belonging to some 'rectangle' $\overline{\Omega}_P = \{|s_1| \le b_1, |s_2| \le b_2\}$

whose sides are parallel to the coordinate axes of the generalized coordinate system (2). The intervals of change of the angle θ in which amplitude DP
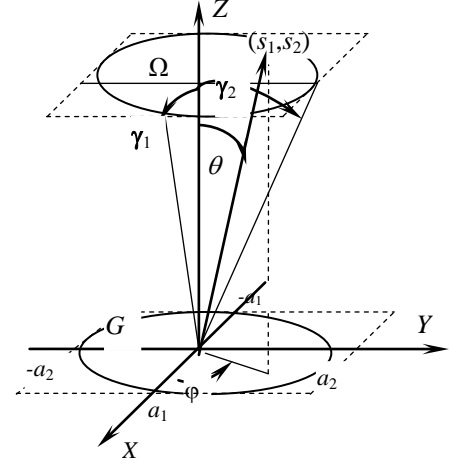

FIG. 1 TO STATEMENT OF THE SYNTHESIS PROBLEM OF FLAT APERTURE

$F(s_1, s_2)$ is given at $\varphi = 0$ $\varphi = \pi/2$ we denote as γ_1 and γ_2, respectively. The parameters c_1 and c_2 characterize the dimensions (in wave lengths) of antenna (aperture) and domain (solid angle) $\overline{\Omega}$, where amplitude DP is given.

Using new variables, (1) takes the form

$$f(s_1, s_2) = AU \equiv \iint\limits_{\overline{G}} U(x, y) e^{i(c_1 x s_1 + c_2 y s_2)} dxdy , \quad (4)$$

which we shall consider as the action of linear integral operator A from complex space of square integrable functions $H_U = L_2(G)$, describing the distribution of the oscillating velocity, into complex space of square integrable functions $H_f = L_2(\mathbb{R}^2)$. The properties of the operator A are defined by the type and geometry of the radiating system.

We define scalar products and norms in the spaces H_U and H_f as follows:

$$(U_1, U_2)_{H_U} = \frac{(2\pi)^2}{c_1 c_2} \iint\limits_{\overline{G}} U^{(1)}(x, y) \cdot \overline{U^{(2)}(x, y)} \, dx \, dy ,$$

$$\|U\|_{H_U} = (U, U)_{H_U}^{1/2} , \quad (5)$$

$$(f_1, f_2)_{H_f} = \iint\limits_{\mathbb{R}^2} f_1(s_1, s_2) \cdot \overline{f_2(s_1, s_2)} \, ds_1 ds_2 ,$$

$$\|f\|_{H_f} = (f, f)_{H_f}^{1/2} . \quad (6)$$

For representation (4) the Percival equality is valid [5]:

$$\| AU \|_{H_f}^2 = \| U \|_{H_U}^2 . \qquad (7)$$

From here follows the property of isometricity of the operator A. On the basis of introduced scalar products and relation $(A\boldsymbol{U}, \boldsymbol{f})_{H_f} = (\boldsymbol{U}, A*\boldsymbol{f})_{H_U}$ we obtain, necessary later on, the expression for conjugate operator

$$A^* f = \iint_{\bar{\Omega}} f(s_1, s_2) e^{-i(c_1 x s_1 + c_2 y s_2)} ds_1 ds_2 . \qquad (8)$$

Consider the synthesis problem of given amplitude DP $F(s_1, s_2)$ in the domain $\bar{\Omega}$ as a minimization problem of the functional

$$\sigma_F(U) = \iint_{\bar{\Omega}} \left\{ \Big[F(s_1, s_2) - |f(s_1, s_2)| \Big]^2 \right\} ds_1 ds_2 +$$
$$+ \iint_{\mathbb{R}^2 \setminus \bar{\Omega}} |f(s_1, s_2)|^2 ds_1 ds_2 \qquad (9)$$

in the space $H_U = L_2(G)$. Here the first summand describes the mean-square deviation of the modulus given and synthesized DP in the domain $\bar{\Omega}$. The second summand imposes constraints on the level of energy emitted outside given solid angle (the domain $\bar{\Omega}$).

Differentiating functional $\sigma_F(U)$ by Hato and using the necessary condition of minimum functional [6] $\mathrm{grad}\,\sigma_F(U) = 0$, we write equation with respect to the function of distribution of the oscillating velocity in an aperture:

$$U(x, y) = \frac{c_1 c_2}{(2\pi)^2} \iint_{\bar{\Omega}} F(s_1, s_2) \exp\big(-i(c_1 x s_1 + c_2 y s_2)\big) \times$$
$$\times \exp\left(i \arg \iint_G U(x', y') e^{i(c_1 x' s_1 + c_2 y' s_2)} dx' dy' \right) ds_1 \, ds_2 . \quad (10)$$

The operator form reads:

$$U = \frac{c_1 c_2}{4\pi^2} A^* \big(F \cdot \exp(i \arg AU) \big). \qquad (11)$$

Acting on both sides of this equation by the operator A and taking into account (4) we obtain the equation with respect to the synthesized DP:

$$f(s_1, s_2) = Bf \equiv \iint_{\bar{\Omega}} F(s_1', s_2') K(s_1, s_2, s_1', s_2'; c_1, c_2) e^{i \arg f(s_1', s_2')} ds_1' ds_2' , \quad (12)$$

where

$$K(Q, Q', \mathbf{c}) = \frac{c_1 c_2}{(2\pi)^2} \iint_{\bar{G}} \exp\Big[i\big(c_1 x(s_1' - s_1) + c_2 y(s_2' - s_2)\big) \Big] dx dy \quad (13)$$

is a kernel essentially dependent on the form of the domain $\bar{G}$.

In the case of symmetric domains $\bar{G}$ the kernel (13) can be simplified. In particular, if the domain $\bar{G}$ has two axes of symmetry, and its upper and lower limits are described, respectively, by the functions $y = \pm \eta(x)$ at $x \in [-1, 1]$, (13) is real and it takes the form

$$K(Q, Q', \mathbf{c}) = \frac{c_1}{2\pi^2} \int_{-1}^{1} \cos\big(c_1 x(s_1' - s_1)\big) \frac{\sin\big(c_2(s_2' - s_2)\eta(x)\big)}{(s_2' - s_2)} dx .$$
$$(14)$$

In the case of a rectangular aperture (13) has the form

$$K(s_1, s_2, s_1', s_2'; c_1, c_2) = \frac{\sin c_1(s_1' - s_1)}{\pi(s_1' - s_1)} \cdot \frac{\sin c_2(s_2' - s_2)}{\pi(s_2' - s_2)} . (15)$$

Note, the Equations (10) and (12) are equivalent in this sense [3]: between solutions of these equations there exists bijection, that to every solution of (10) corresponds the solution of (12) and on the contrary. If f_* is the solution of (12), then corresponding solution U_* of (10) is determined by the formula

$$U_*(x, y) = \frac{c_1 c_2}{(2\pi)^2} \iint_{\bar{\Omega}} F(Q) \exp\Big[i\big(\arg f_*(Q) - (c_1 x s_1 + c_2 y s_2)\big) \Big] ds_1 ds_2 ; (16)$$

if U_* is the solution of (10), then corresponding solution f_* of (12) is determined by (4).

Using a more general expression (13) for $K(Q, Q', \mathbf{c})$, we consider the operator

$$Df \equiv \iint_{\Omega} K(Q, Q', \mathbf{c}) f(Q') dQ' \qquad (17)$$

and corresponding to it quadratic form

$$(Df, f) = \iint_{\bar{\Omega}} \iint_{\bar{\Omega}} K(Q, Q', \mathbf{c}) f(Q') dQ' \overline{f(Q)} dQ =$$

$$= \frac{c_1 c_2}{(2\pi)^2} \iint_{\bar{G}} \left| \iint_{\bar{\Omega}} f(Q) \exp(i\mathbf{c}(P, Q)) dQ \right|^2 dx dy > 0 . \qquad (18)$$

From here follows that $K(Q, Q', \mathbf{c})$ is a positive kernel [7]. Accordingly the operator D is also positive on the nonnegative functions cone $\mathcal{K}$ of the space $C(\bar{G})$ [8]. Based on this property the operator D retains invariant cone $\mathcal{K}$, that is $D\mathcal{K} \subset \mathcal{K}$.

Since a set of values of operator A is a set of continuous functions, belonging to the space $L_2(\Omega)$,

and a set of continuous functions in the domain $\overline{\Omega}$, is dense in the space $L_2(\Omega)$ [7], we shall investigate the solutions of (12) in the complex space $\mathbf{C}(\overline{\Omega})$.

On the basis of decomplexification [9] we consider the complex space $\mathbf{C}(\overline{\Omega})$ as a direct sum $\mathbf{C}(\overline{\Omega}) = C(\overline{\Omega}) \oplus C(\overline{\Omega})$ of two real spaces of continuous functions in the domain $\overline{\Omega}$. The elements of this space have the form whose elements are given in the form: $f = (u,v)^\top \in \mathbf{C}(\overline{\Omega})$, $u = \mathrm{Re}(f) \in C(\overline{\Omega})$, $v = \mathrm{Im}(f) \in C(\overline{\Omega})$. Norms in these spaces we shall introduce as:

$$\|u\|_{C(\overline{\Omega})} = \max_{Q \in \overline{\Omega}} |u(Q)|, \quad \|v\|_{C(\overline{\Omega})} = \max_{Q \in \overline{\Omega}} |v(Q)|,$$

$$\|f\|_{\mathbf{C}(\overline{\Omega})} = \max\left(\|u\|_{C(\overline{\Omega})}, \|v\|_{C(\overline{\Omega})}\right). \tag{19}$$

The Equation (12) in decomplexified space $\mathbb{C}(\overline{\Omega})$ we reduce to equivalent to it system of equations

$$u(Q) = B_1(u,v) \equiv \iint_{\overline{\Omega}} F(Q')K(Q,Q',\mathbf{c}) \frac{u(Q')}{\sqrt{u^2(Q')+v^2(Q')}} dQ',$$

$$v(Q) = B_2(u,v) \equiv \iint_{\overline{\Omega}} F(Q')K(Q,Q',\mathbf{c}) \frac{v(Q')}{\sqrt{u^2(Q')+v^2(Q')}} dQ'.$$

$$\tag{20}$$

Denote a closed convex set of continuous functions as $S_M \subset \mathbb{C}(\overline{\Omega})$ supposing that

$$S_M = S_{M_u} \oplus S_{M_v}, \qquad S_{M_u} = \left\{ u \in S_{M_u} : \|u\|_{C(\overline{\Omega})} \leq M \right\},$$

$$S_{M_v} = \left\{ v \in S_{M_v} : \|v\|_{C(\overline{\Omega})} \leq M \right\},$$

$$M = \max_{Q \in \overline{\Omega}} \iint_{\overline{\Omega}} F(Q') |K(Q,Q',\mathbf{c})| dQ.$$

Consider one of the properties of the function $\exp(i \arg f(Q'))$ included in (12) at $f(Q') \to 0$. It is obviously that

$$\exp(i \arg f(Q')) = \frac{f(Q')}{|f(Q')|} \equiv \frac{u(Q')+iv(Q')}{\left(u^2(Q')+v^2(Q')\right)^{1/2}} \tag{21}$$

is continuous function if $u(Q') = \mathrm{Re}\, f(Q')$ and $v(Q') = \mathrm{Im}\, f(Q')$ are continuous functions, where $|\exp(i \arg f(Q'))| = 1$ for any $f(Q')$. If $u(Q') \to 0$ and $v(Q') \to 0$ simultaneously, then $f(Q') \equiv 0$ is a complex zero, the argument of which is indefinite according to definition [10, p. 20]. On this basis we redefine

$\exp(i \arg f(Q'))$ at $u(Q') \to 0$ and $v(Q') \to 0$ as a function which module is equal to unit and indefinite argument.

Theorem 1. *The operator $\mathbf{B} = (B_1, B_2)^T$ determined by (20) maps a closed convex set S_M of the Banach space $\mathbb{C}(\overline{\Omega})$ in itself and it is completely continuous.*

To prove the theorem it is necessary to show that $\mathbf{B} : \mathbf{C}(\overline{\Omega}) \to \mathbf{C}(\overline{\Omega})$ and using the Arzela Theorem [6] to prove that functions of a set $S_g = \mathbf{B}S_M$ are uniformly bounded and equapotentially continuous where the inequality $\mathbf{B}S_M \subset S_M$ holds.

As a **corollary** of Theorem 1 the satisfaction of the Schauder principle conditions [11] in accordance to which the operator $\mathbf{B} = (B_1, B_2)^T$ has a fixed point $f_* = (u_*, v_*)^T$, belonging to a set S_M, follows. This point solves (20) and, respectively, (12). Substituting $f_* = (u_*, v_*)^T$ into (16), we obtain the solution of (10), which is a stationary point of functional (9).

Concerning the synthesis problems of linear radiator for the case of one-dimensional domains $\overline{\Omega}$, solutions of a system of equations similar to (20) were investigated, in particular, in [2, 3]. The obtained there results show that nonuniqueness and branching of solutions dependent on the size of one-dimensional physical parameter of the problem, are characteristic for equations of the type (12). The results [2, 3] can not be transferred directly to two-dimensional nonlinear integral equations (12), (20). Here, unlike the branching points [3], there exist branching lines of solutions, and the problem of finding the branching lines is a nonlinear two-parameter spectral problem.

We shall formulate important properties of (12), necessary further, which are checked directly.

1°. If the function $f(Q)$ solves (12), then complex-conjugate function $\overline{f(Q)}$ is also a solution of (12).

2°. If the function $f(Q)$ solves (12), then $e^{i\tau} f(Q)$ is also a solution of (12), where τ is an arbitrary real constant.

3°. For functions $F(s_1, s_2)$ even on two arguments (or on one argument) linear operator $\mathbf{B}$, that is in the right part of (12), is invariant concerning the type of parity function $\arg f(s_1, s_2)$ on two arguments (or concerning the argument on which the function $F(s_1, s_2)$ is even).

Further for the uniqueness of desired solutions using the property 2° we shall select the parameter τ to fulfill the equality

$$\arg f(0,0) = 0. \tag{22}$$

It is easy to check that for the case of a symmetric domain $\bar{\Omega}$ the function

$$f_0(Q,\mathbf{c}) = \iint\limits_{\bar{\Omega}} F(Q')K(Q,Q',\mathbf{c})\,dQ' \tag{23}$$

is one of the solutions of (12). Further we shall call solution (23) as primary. Since, as shown above, the operator D defined by equality (17) is positive on the cone of nonnegative functions $\mathcal{K} \in \mathbb{C}(\bar{\Omega})$ and $F \subset \mathcal{K}$, then $f_0 = DF$ is also nonnegative function in the domain $\bar{\Omega}$.

To find the branching lines and complex solutions of (12), which branch-off from the real solution $f_0(Q,\mathbf{c})$ we consider the problem on finding such a set of values of parameters $\mathbf{c}^{(0)} = (c_1^{(0)}, c_2^{(0)})$ and all distinct from $f_0(Q,\mathbf{c})$ solutions of (20), which satisfy the conditions

$$\max_{Q\in\bar{\Omega}}\left|u(Q,\mathbf{c}) - f(Q,\mathbf{c}^{(0)})\right| \to 0, \ \max_{Q\in\bar{\Omega}}\left|v(Q,\mathbf{c})\right| \to 0 \tag{24}$$

as $\left|\mathbf{c} - \mathbf{c}^{(0)}\right| \to 0$.

These conditions mean that it is necessary to find small continuous in $\bar{\Omega}$ solutions

$$w(Q,\mathbf{c}) = u(Q,\mathbf{c}) - f_0(Q,\mathbf{c}^{(0)}), \qquad \omega(Q,\mathbf{c}) = v(Q,\mathbf{c}),$$

which converge uniformly to zero at $\mathbf{c} \to \mathbf{c}^{(0)}$. In addition, it is necessary also to take into account the direction of convergence of vector $\mathbf{c}$ to $\mathbf{c}^{(0)}$.

Put

$$c_1 = c_1^{(0)} + \mu, \ \ c_2 = c_2^{(0)} + \nu \tag{25}$$

and we shall find the desired solutions in the form

$$u(Q,\mathbf{c}) = f_0(Q,\mathbf{c}^{(0)}) + w(Q,\mu,\nu), \ v(Q,\mathbf{c}) = \omega(Q,\mu,\nu). \tag{26}$$

Further we omit the dependence of functions $w(Q,\mu,\nu)$ and $\omega(Q,\mu,\nu)$ on parameters μ and ν to simplify the notations.

Indicate the properties of integrands in the system (20): they are continuous functions of arguments. Substituting (25) and (26) in (20), we expand integrands in the uniformly convergent power series with respect to the functional arguments on functional

arguments w, ω and numerical parameters μ and ν in the neighborhood of point $(\mathbf{c}^{(0)}, f_0(Q,\mathbf{c}^{(0)}),0)$:

$$F(Q')K(Q,Q',\mathbf{c})\frac{u(Q')}{\sqrt{u^2(Q') + v^2(Q')}} =$$

$$= \sum_{m+n+p+q\geq 0} A_{mnpq}(Q,Q',\mathbf{c}^{(0)})w^m(Q')\omega^n(Q')\mu^p\nu^q, \tag{27}$$

$$F(Q')K(Q,Q',\mathbf{c})\frac{v(Q')}{\sqrt{u^2(Q') + v^2(Q')}} =$$

$$= \sum_{m+n+p+q\geq 1} B_{mnpq}(Q,Q',\mathbf{c}^{(0)})w^m(Q')\omega^n(Q')\mu^p\nu^q. \tag{28}$$

Here $A_{mnpq}(Q,Q',\mathbf{c}^{(0)})$, $B_{mnpq}(Q,Q',\mathbf{c}^{(0)})$ are the coefficients of expansion continuously dependent on arguments. Substituting (25) - (27) in (20) and considering $f_0(Q',\mathbf{c}^{(0)})$ as a solution of (20), we obtain a system of nonlinear equations with respect to small solutions w and ω:

$$w(Q) = a_{10}(Q,\mathbf{c}^{(0)})\mu + a_{01}(Q,\mathbf{c}^{(0)})\nu +$$

$$+ \sum_{m+n+p+q\geq 2} \mu^p\nu^q \iint\limits_{\bar{\Omega}} A_{mnpq}(Q,Q',\mathbf{c}^{(0)})w^m(Q')\omega^n(Q')\,dQ', \tag{29}$$

$$\omega(Q) - \iint\limits_{\bar{\Omega}} F(Q)K(Q,Q',\mathbf{c}^{(0)})\frac{\omega(Q')}{f_0(Q',\mathbf{c}^{(0)})}\,dQ' =$$

$$= \sum_{m+n+p+q\geq 2} \mu^p\nu^q \iint\limits_{\bar{\Omega}} B_{mnpq}(Q,Q',\mathbf{c}^{(0)})w^m(Q')\omega^n(Q')\,d\Omega_{Q'},$$

$$\tag{30}$$

where

$$a_{10}(Q,\mathbf{c}^{(0)}) = \iint\limits_{\bar{\Omega}} A_{0010}(Q,Q',\mathbf{c}^{(0)})\,dQ' \qquad\qquad ,$$

$$a_{01}(Q,\mathbf{c}^{(0)}) = \iint\limits_{\bar{\Omega}} A_{0001}(Q,Q',\mathbf{c}^{(0)})\,dQ'.$$

For further application of the methods of the branching theory of solutions of nonlinear equations [12] to system (29) and (30), it is necessary to find solutions distinct from the trivial of linear homogeneous integral equation. We obtain this equality equating to zero the left part of (30):

$$\varphi(Q) = T(c_1,c_2)\varphi \equiv \iint\limits_{\bar{\Omega}} \frac{F(Q')}{f_0(Q',c_1,c_2)}K(Q,Q',c_1,c_2)\varphi(Q')\,dQ' \tag{31}$$

at $f_0(Q',\mathbf{c}) > 0$. Note, that the operator $T(\mathbf{c}): \mathbf{C}(\bar{\Omega}) \to \mathbf{C}(\bar{\Omega})$ is completely continuous.

Accordingly to [12] those values of parameters $(c_1^{(0)}, c_2^{(0)}) \in \mathbb{R}^2$ at which the linear homogeneous

equation (31) has solutions distinct from identical zero are points of possible branching of solutions of nonlinear equations (29) and (30). Eigenfunctions of (31) are used at construction of branching-off solutions of (29) and (30).

Note, in a general case (31) is a nonlinear two-parameter spectral problem.

Implicit Function Method for Finding the Connected Components of Spectrum of Nonlinear Two-Parameter Spectral Problems

Here we shall consider a method for solving one class of nonlinear two-parameter spectral problems, which include, in particular, the problem on finding the solutions of linear homogeneous integral Equation (31). Since the proposed approach can be applied to finding not only the connected components of the spectrum of integral equations, we present the material on the general (operational) level.

Let E and V be a complex Banach space and a vector parameter $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ belongs to the domain (open connected set) $\Lambda = \Lambda_1 \times \Lambda_2$ of the complex space $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. Here $\lambda_i \in \Lambda_i \subset \mathbb{C}$, $\Lambda_i = \{\lambda_i \in \Lambda_i : |\lambda_i| < r_\lambda\}$ $(i = 1, 2)$, r_λ is some real constant. Consider the operator-function $\mathsf{A}(\cdot, \cdot) : \Lambda \to \mathsf{L}(E, V)$, where to each $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \Lambda$ the operator $\mathsf{A}(\lambda_1, \lambda_2) \in \mathsf{L}(E, V)$ is assigned. Here $\mathsf{L}(E, V)$ is the space of linear bounded operators [9].

Let us consider the nonlinear two-parameter spectral problem of the form

$$\mathsf{A}(\lambda_1, \lambda_2)x = 0, \tag{32}$$

in which it is necessary to find the eigenvalues $\boldsymbol{\lambda} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}\right) \in \Lambda$ and corresponding to them eigenvectors $x^{(0)} \in E$ ($x^{(0)} \neq 0$) such that $\mathsf{A}\left(\lambda_1^{(0)}, \lambda_2^{(0)}\right)x^{(0)} = 0$.

Without limiting the generality, in view on the form of (31), we shall present the operator-function $\mathsf{A}(\lambda_1, \lambda_2)$ as

$$\mathsf{A}(\lambda_1, \lambda_2) = T(\lambda_1, \lambda_2) - I. \tag{33}$$

Here $T(\lambda_1, \lambda_2)$ is a linear completely continuous operator acting in the Banach space E and analytically dependent on two-dimensional parameter (λ_1, λ_2), I

is a unique operator in E. In this case we put that $\mathsf{A}(\cdot, \cdot) : \Lambda \to \mathsf{L}(E, E)$, i. e. $V = E$, and $A(\lambda_1, \lambda_2) \in \mathsf{L}(E, E)$.

Let the Banach spaces E and E_n $(n = 1, 2, \ldots)$ and also the system $\mathsf{P} = (p_n)_{n \in \mathbb{N}}$ of linear bounded operators $p_n : E \to E_n$ such that

$$\|p_n x\|_{E_n} \to \|x\|_E \qquad (n \in \mathbb{N}) \qquad \forall x \in E \tag{34}$$

be given. Operators p_n are called conjunctive operators [13]. From the principle of uniform boundedness for p_n the inequality $\|p_n\| \leq \mathrm{const}$ $(n \in \mathbb{N})$ follows.

Let in every space E_n the element x_n be selected. Writing these elements in order to increase the numbers we shall form a sequence $\{x_n\}$.

Definition 1 [13]. *The sequence* $\{x_n\}_{n \in \mathbb{N}'}$, *from* $x_n \in E_n$ P *converges to* $x \in E$, *if* $\|x_n - p_n x\|_{E_n} \to 0$ ($n \in \mathbb{N}'$); *we denote* $x_n \xrightarrow{\mathsf{P}} x$ ($n \in \mathbb{N}'$).

As noted above, in this definition the elements x and x_n belong to various spaces. The properties of P-convergence are given, in particular, in [13]. At application of that or other approach to discretization of initial problem the operator-function $\mathsf{A}(\lambda_1, \lambda_2)$ is approximated, respectively, by the approximate operator-functions $\mathsf{A}_n(\lambda_1, \lambda_2)$, $n \in \mathbb{N}$. As a result, for each $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \Lambda$ we obtain a sequence of operators $A_n \in \mathsf{L}(E_n, E_n)$ converging to operator $A \in \mathsf{L}(E, E)$ at satisfaction of the corresponding conditions. We shall present the definition of convergence of the operators $A_n \in \mathsf{L}(E_n, V_n)$ to $A \in \mathsf{L}(E, V)$ necessary later on. Let the Banach spaces E, V, E_n, V_n, $n = 1, 2, \ldots$, and linear systems $\mathsf{P} = (p_n)$ and $Q = (q_n)$ of linear (connected) operators $p_n : E \to E_n$ and $q_n : V \to V_n$ such that

$$\|p_n x\| \to \|x\| \text{ and } \|q_n y\| \to \|y\| \quad (n \in \mathbb{N}) \ \forall x \in E, \ y \in V \tag{35}$$

be given.

Definition 2 [13]. *The sequence* $\{A_n\}_{n \in \mathbb{N}}$ *of the operators* $A_n : E_n \to V_n$ $\mathsf{P}Q$- *converges* (*or discretely converges*) *to the operator* $A : E \to V$, *if for each* P - *convergence sequence* $\{x_n\}$ *the following relation is valid:*

$$x_n \xrightarrow{\;P\;} x \;\; (n \in \mathbb{N}) \;\; \Rightarrow \;\; A_n x_n \xrightarrow{\;Q\;} Ax; \qquad (36)$$

we denote $A_n \xrightarrow{\;PQ\;} A \;\; (n \in \mathbb{N})$ *or* $A_n \to A \;\; (n \in \mathbb{N})$.

Discretization of initial problem (32), the choice of the spaces E_n and determination of operators $p_n : E \to E_n$, $q_n : V \to V_n$ are realized by various ways. In particular, if the operator-function is described by (33) and E is the separable (infinite-dimensional) Hilbertian space, one of the approaches to discretization (32) consists in the following. Consider an arbitrary complete orthonormal in E system of functions $\{x_k\}_{k=1}^{\infty}$. Each element $x \in E$ can be presented as a series $x = \sum\limits_{k=1}^{\infty} c_k x_k$.

Here $c_k = (x, x_k)$ is the Fourier coefficient of the element x. If $T(\lambda_1, \lambda_2)$ is a linear continuous operator, acting in the separable Hilbertian space, then matrix representation has the form [14]:

$$T_M(\lambda_1, \lambda_2) = \left(t_{jk}(\lambda_1, \lambda_2) \right)_{j,k=1}^{\infty}. \qquad (37)$$

Here $t_{jk}(\lambda_1, \lambda_2) = \left(T(\lambda_1, \lambda_2) x_k, x_j \right)$. This sequence of Fourier coefficients of the element $y = T(\lambda_1, \lambda_2) x$ is derived from the sequence of Fourier coefficients of the element x by means of transformation by the matrix $T_M(\lambda_1, \lambda_2)$.

Using the matrix representation of operator $T_M(\lambda_1, \lambda_2)$ the spectral problem (32) is formulated as

$$\mathsf{A}_M(\lambda_1, \lambda_2)x \equiv \left(T_M(\lambda_1, \lambda_2) - I_M \right)x = 0, \qquad (38)$$

where I_M is a unit matrix in the space of sequences l_2. Thus, the operators $T(\lambda)$ and $T_M(\lambda)$ are equivalent in the sense that they assign one and the same element $y \in E$ to the same element $x \in E$. But we obtain the Fourier coefficients of element $y = T(\lambda)x$ as a result of action of the operator $T_M(\lambda)$ on the element x.

Obviously, that the spectrums of these operators coincide, that is the spectral problems (32) and (38) are equivalent.

In this case we shall put that the finite dimensional spaces E_n are generated by the bases $\{x_k\}_{k=1}^{n}$ $(n \in \mathbb{N})$ and to each element $x \in E$ the operators $p_n : E \to E_n$ assign the element $x = \sum\limits_{k=1}^{n} c_k x_k$, where $c_k = (x, x_k)$. As a result, the approximated operator $T_{M_n}(\lambda)$ is described by the finite-dimensional matrix-function

$$T_{M_n}(\lambda_1, \lambda_2) = \left(t_{jk}(\lambda_1, \lambda_2) \right)_{j,k=1}^{n}. \qquad (39)$$

Applying other methods of discretization to problem (32), in particular, the quadrature (cubature) processes for the case of homogeneous integral equations and change of derivates by their difference analogues in differential equations, we obtain approximate problems to find approximately the eigenvalues and eigenfunctions in the form

$$\mathsf{A}_n(\lambda_1, \lambda_2)x_n = 0, \quad n \in \mathbb{N}. \qquad (40)$$

Moreover, the problem of determination the eigenvalues is reduced to finding the roots of the determinant of n-th order that is the roots of equation

$$\Psi_n(\lambda_1, \lambda_2) \equiv \det \begin{pmatrix} a_{11}(\lambda_1, \lambda_2) & a_{12}(\lambda_1, \lambda_2) & ... & a_{1n}(\lambda_1, \lambda_2) \\ a_{21}(\lambda_1, \lambda_2) & a_{22}(\lambda_1, \lambda_2) & ... & a_{2n}(\lambda_1, \lambda_2) \\ .... & .... & ... & .... \\ a_{n1}(\lambda_1, \lambda_2) & a_{n2}(\lambda_1, \lambda_2) & ... & a_{nn}(\lambda_1, \lambda_2) \end{pmatrix} = 0$$

$$(n \in \mathbb{N}). \qquad (41)$$

Note, if the coefficients $a_{ij}(\lambda_1, \lambda_2)$ are continuously differentiable functions of arguments, the partial derivatives of the function $\Psi_n(\lambda_1, \lambda_2)$ are found by the rules of determinant derivation:

$$\frac{\partial \Psi_n(\lambda_1, \lambda_2)}{\partial \lambda_i} =$$

$$= \sum_{j=1}^{n} \det \begin{pmatrix} a_{11}(\lambda_1, \lambda_2) & \cdots & a_{1,j-1}(\lambda_1, \lambda_2) & \dfrac{\partial a_{1,j}(\lambda_1, \lambda_2)}{\partial \lambda_i} & a_{1,j+1}(\lambda_1, \lambda_2) & \cdots & a_{1,n}(\lambda_1, \lambda_2) \\ a_{21}(\lambda_1, \lambda_2) & \cdots & a_{2,j-1}(\lambda_1, \lambda_2) & \dfrac{\partial a_{2,j}(\lambda_1, \lambda_2)}{\partial \lambda_i} & a_{2,j+1}(\lambda_1, \lambda_2) & \cdots & a_{2,n}(\lambda_1, \lambda_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}(\lambda_1, \lambda_2) & \cdots & a_{n,j-1}(\lambda_1, \lambda_2) & \dfrac{\partial a_{n,j}(\lambda_1, \lambda_2)}{\partial \lambda_i} & a_{n,j+1}(\lambda_1, \lambda_2) & \cdots & a_{n,n}(\lambda_1, \lambda_2) \end{pmatrix},$$

$$i = 1, 2. \qquad (42)$$

Consider the auxiliary nonlinear one-parameter spectral problem, necessary later on, as a special case of (32). Assume that variable λ_2 in the operator-function $\mathsf{A}(\lambda_1, \lambda_2)$ is expressed by some one-valued differentiated function $\lambda_2 = z(\lambda_1)$ mapping the subdomain $\Lambda_{1,\beta} \subset \Lambda_1$ into some subdomain $\Lambda_{2,\beta} \subset \Lambda_2$. In the simplest case we put $\lambda_2 = \beta\lambda_1$ (β is some real parameter). Introduce the operator-function $\mathsf{A}_\beta(\lambda_1) \equiv \mathsf{A}(\lambda_1, z(\lambda_1))$ at $\lambda_1 \in \Lambda_{1,\beta}$ (reduction of the

operator-function $A(\lambda_1, \lambda_2)$) with which is connected nonlinear one-dimensional spectral problem

$$A_\beta(\lambda_1)x = 0. \tag{43}$$

Here we assign to each value $\lambda = (\lambda_1, z(\lambda_1)) \in \Lambda$ the operator $A_\beta(\lambda_1, z(\lambda_1)) \in L(E, E)$.

Analogously to (40), we consider an approximate sequence of discrete problem of (43)

$$A_{\beta,n}(\lambda_1, z(\lambda_1))x_n = 0, \quad n \in \mathbb{N}. \tag{44}$$

Denote the spectrum of operator-function $A_\beta(\lambda_1)$ as $s(A_\beta)$. Assume that $s(A_\beta) \neq \Lambda_{1,\beta}$. For the spectrum $s(A)$ of the problem (34) the following theorem is valid.

Theorem 2. *Let the following conditions be satisfied:*

1) *operator-function $A(\cdot, \cdot): \Lambda \rightarrow \in L(E, E)$ is holomorphic, and $s(A) \neq \Lambda$;*

2) *operator-functions $A_n(\cdot, \cdot): \Lambda \rightarrow \in L(E_n, E_n)$ are holomorphic and for any closed bounded set $\Lambda_0 \subset \Lambda$ the following inequality $\max_{\lambda \in \Lambda_0} \|A_n(\lambda_1, \lambda_2)\| \leq$*
$$\leq c(\Lambda_0) = \text{const} \quad (n \in \mathbb{N}) \text{ is valid;}$$

3) *operators $A(\lambda_1, \lambda_2) \in L(E, E)$, $A_n(\lambda_1, \lambda_2) \in \in L(E_n, E_n)$ ($n \in \mathbb{N}$) are the Fredholm operators with zero index for any $\lambda = (\lambda_1, \lambda_2) \in \Lambda$;*

4) *spectrum $s(A_\beta) \neq \Lambda_{1,\beta}$ and a sequence of functions $\Psi_n(\lambda_1, \lambda_2)$ are differentiable in the domain Λ;*

5) *$A_n(\lambda) \rightarrow A(\lambda)$ is stable for any $\lambda \in r(A) = \Lambda \setminus s(A)$.*

Then the following statements are true:

1) *every point of spectrum $\lambda_1^{(0)} \in s(A_\beta)$ is isolated, it is eigenvalue of the operator $A_\beta(\lambda_1) \equiv A(\lambda_1, z(\lambda_1))$, the finite-dimensional eigensubspace $N(A(\lambda_1^{(0)}))$ and the finite-dimensional root subspace correspond to it;*

2) *for each $\lambda_1^{(0)} \in s(A_\beta)$ there exists a sequence $\{\lambda_{1,n}^{(0)}\}$ from $\lambda_{1,n}^{(0)} \in s(A_{\beta,n})$ ($n > n_0$), such that $\lambda_{1,n}^{(0)} \rightarrow \lambda_1^0$;*

3) *each point $\lambda^{(0)} = (\lambda_1^{(0)}, z(\lambda_1^{(0)})) \in \Lambda$ is a spectrum point of the operator-function $A(\lambda_1, \lambda_2)$;*

4) *if in some small ε_0 - neighborhood of the point $\lambda^{(0)} = (\lambda_1^{(0)}, z(\lambda_1^{(0)})) \in \Lambda$ for all n, larger than some number N_0, the sequence of partial derivatives*

$$\left\{\frac{\partial \Psi_n}{\partial \lambda_2}(\lambda_{1,n}^0, z(\lambda_{1,n}^0))\right\}$$ *is nonzero, then in an arbitrarily small ε_*-neighborhood of the point $(\lambda_1^{(0)}, z(\lambda_1^{(0)})) \in \Lambda$ there exists a continuously differentiable function $\lambda_{2,N_*} = \varphi_{N_*}(\lambda_1)$ which is a solution of (41), where*
$$\lambda_{2,N_*}^{(0)} = \varphi_{N_*}(\lambda_{1,N_*}^{(0)}) \quad \text{and} \quad \text{at} \quad \text{the} \quad \text{point}$$
$(\lambda_{1,N_*}^{(0)}, \lambda_{2,N_*}^{(0)}) = (\lambda_{1,N_*}^{(0)}, \varphi_{N_*}(\lambda_{1,N_*}^{(0)}))$ *it arbitrary small deviates from the point spectrum of auxiliary one-parameter problem (43) $|\lambda_{1,N_*}^{(0)} - \lambda_1^{(0)}| < \varepsilon_*$. That is, in some bicylindrical domain $\Lambda_0 = \{(\lambda_1, \lambda_2) \in \Lambda_0 : |\lambda_1 - \lambda_1^{(0)}| < \varepsilon_1, |\lambda_2 - \lambda_2^{(0)}| < \varepsilon_2\}$ there exists a connected component of spectrum of the operator-function $A_{N_*}(\lambda_1, \lambda_2)$ (ε_1, ε_2 are small real constants).*

Proof. The proof of Theorem is based on Theorems 1, 2 [13, pp. 68, 69] and on existence of implicit functions (see, for example, [15]). At first we show that the conditions of Theorem 1 [13, p. 68] concerning the existence of discrete spectrum of operator-function $A_\beta(\lambda_1)$, follow from the conditions of formulated Theorem. Under the conditions of Theorem the operator $A(\lambda_1, \lambda_2)$ is Fredholm operator with zero index for each $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, and the operator-function $A(\cdot, \cdot): \Lambda \rightarrow \in L(E, E)$ is holomorphic. From here follows that for each $\lambda_1 \in \Lambda_{1,\beta}$ the operator $A_\beta(\lambda_1)$, as reduction of the operator $A(\lambda_1, \lambda_2)$, is also Fredholm operator with zero index, and the operator-function $A_\beta(\lambda_1): \Lambda_{1,\beta} \rightarrow L(E, E)$ is holomorphic. So, for the operator-function $A_\beta(\lambda_1)$ the conditions of Theorem 1 [13] are satisfied. From Theorem 1 follows: each point $\lambda_1^{(0)} \in s(A_\beta)$ is isolated, it is the eigenvalue of the operator $A_\beta(\lambda_1)$, the finite-dimensional eigensubspace and finite dimensional root subspace corresponds to it. Thus, each point $\lambda^{(0)} = (\lambda_1^{(0)}, z(\lambda_1^{(0)})) \in \Lambda$ is the spectrum point of the operator-function $A(\lambda_1, \lambda_2)$.

In addition, the conditions of Theorem 2 [13] are satisfied for the operator-function $A_\beta(\lambda_1): \Lambda_{1,\beta} \rightarrow L(E, E)$. From this theorem follows: at n larger than some $n_0 \in \mathbb{N}$ for each $\lambda_1^{(0)} \in s(A_\beta)$ there exists a sequence $\{\lambda_{1,n}\}$ such that $\lambda_{1,n} \in s(A_{\beta,n})$ and $\lambda_{1,n} \rightarrow \lambda_1^{(0)}$ as $n \rightarrow \infty$. Thus, each point $(\lambda_1^{(0)}, \lambda_2^{(0)}) =$

$= \left(\lambda_1^{(0)}, z\left(\lambda_1^{(0)} \right) \right)$ is the eigenvalue of operator-function $A_\beta(\lambda_1) : \Lambda_{1,\beta} \to L(E,E)$ and, respectively, the eigenvalue of operator-function $A(\cdot, \cdot) : \Lambda \to \in L(E,E)$. Since $\lambda_{1,n}^{(0)} \in s(A_{\beta,n})$ is the root of (41), then from here follows that for $n \in \mathbb{N}$

$$\Psi_n\left(\lambda_{1,n}^{(0)}, z\left(\lambda_{1,n}^{(0)} \right) \right) \to \Psi_n\left(\lambda_1^{(0)}, z\left(\lambda_1^{(0)} \right) \right) = 0 \quad \text{as } n \to \infty. \quad (45)$$

From the criterion of convergence of sequence $\lambda_{1,n}^{(0)} \in s(A_{\beta,n})$ to $\lambda_1^{(0)} \in s(A_\beta)$ follows that for an arbitrarily small number $\varepsilon_* > 0$ there exists such number $N_* > n_0$ that $\left| \lambda_{1,N_*}^{(0)} - \lambda_1^{(0)} \right| < \varepsilon_*$ and $\Psi_{N_*}\left(\lambda_{1,N_*}^{(0)}, z\left(\lambda_{1,N_*}^{(0)} \right) \right) = 0$.

Let λ_1, λ_2 be independent variables in the domain Λ, and $\left(\lambda_1^{(0)}, \lambda_2^{(0)} \right) = \left(\lambda_1^{(0)}, z\left(\lambda_1^{(0)} \right) \right) \in \Lambda$ be a spectrum point of the operator $A(\lambda_1, \lambda_2)$ belonging to $s(A_\beta) \subset s(A)$. Under the conditions of theorem the functions $\Psi_n(\lambda_1, \lambda_2)$ are differentiable in the neighborhood of the point $\left(\lambda_1^{(0)}, \lambda_2^{(0)} \right)$ and $\frac{\partial \Psi_{N_*}}{\partial \lambda_2}\left(\lambda_{1,N_*}^{(0)}, z\left(\lambda_{1,N_*}^{(0)} \right) \right) \neq 0$. In addition the point $\left(\lambda_{1,N_*}^{(0)}, z\left(\lambda_{1,N_*}^{(0)} \right) \right)$ belongs to ε_*-vicinity of the point $\left(\lambda_1^{(0)}, \lambda_2^{(0)} \right)$. According to the Theorem about implicit function [15] in some neighborhood of point $\left(\lambda_1^{(0)}, \lambda_2^{(0)} \right)$ there exists the continuous differentiable function $\lambda_2 = \varphi_{N_*}(\lambda_1)$, solving the equation (41), where $\varphi_n\left(\lambda_{1,N_*}^{(0)} \right) = z\left(\lambda_{1,N_*}^{(0)} \right)$. From here follows the existence of connected component of spectrum of the operator-function $A(\cdot, \cdot) : \Lambda \to L(E,V)$ in some bicylindrical domain

$$\Lambda_0 = \left\{ (\lambda_1, \lambda_2) \in \Lambda_0 : \left| \lambda_1 - \lambda_1^{(0)} \right| < \varepsilon_1, \left| \lambda_2 - \lambda_2^{(0)} \right| < \varepsilon_2 \right\}.$$

At the point $\left(\lambda_{1,N_*}^{(0)}, \lambda_{2,N_*}^{(0)} \right) = \left(\lambda_{1,N_*}^{(0)}, \varphi_{N_*}\left(\lambda_{1,N_*}^{(0)} \right) \right)$ this component deviates from the point of spectrum $\lambda^{(0)} = \left(\lambda_1^{(0)}, z\left(\lambda_1^{(0)} \right) \right) \in \Lambda$ of operator-function $A(\lambda_1, \lambda_2)$ arbitrarily. Theorem is proved.

Consider Equation (41), putting that the coefficients $a_{ij}(\lambda_1, \lambda_2)$ of the matrix-function $A_n(\lambda_1, \lambda_2)$ are differentiable continuous functions, *i. e.* partial derivatives $\partial \Psi_n(\lambda_1, \lambda_2)/\partial \lambda_1$ and $\partial \Psi_n(\lambda_1, \lambda_2)/\partial \lambda_2$ determined by (42), exist and they are continuous. If at

the points $\left\{ \lambda_{1,\nu}^{(0)}, \lambda_{2,\nu}^{(0)} \right\} \in \Lambda_1 \times \Lambda_2$ the derivative $\partial \Psi_n/\partial \lambda_2$ is nonzero, then solving the Cauchy problem [16]

$$\frac{d\lambda_2}{d\lambda_1} = -\frac{\partial \Psi_n(\lambda_1, \lambda_2)/\partial \lambda_1}{\partial \Psi_n(\lambda_1, \lambda_2)/\partial \lambda_2}, \quad (46)$$

$$\lambda_2\left(\lambda_1^{(\nu)} \right) = \lambda_2^{(\nu)}, \quad (47)$$

in the vicinity of each point $\lambda_1^{(\nu)} \in \Lambda_1$ we find ν-th connected component of the spectrum of the operator function $A_n(\lambda_1, \lambda_2)$.

Numerical Finding of the Branching Lines of Solutions

Return to finding the solutions of (31), where c_1, c_2 are spectral parameters. Let $(c_1, c_2) \in \Lambda_c$, $\Lambda_c = \Lambda_{c_1} \times \Lambda_{c_2}$, where

$$\Lambda_{c_i} = \left\{ c_i \in \Lambda_{c_i} : 0 < c_i < r_c \right\}.$$

We shall present equivalent to the previous definition of eigenvalues and eigenvectors for (31), which later on will be used for finding the numerical solutions of this equation.

Definition 3. *We shall call a set of values of parameters* $\left(c_1^{(0)}, c_2^{(0)} \right) \in \Lambda_c$ *at which* $\mu = 1$

$$\mu\varphi(Q) = \iint_{\bar{\Omega}} \frac{F(Q')}{f_0(Q', c_1, c_2)} K(Q, Q', c_1, c_2)\varphi(Q')dQ', \quad (48)$$

as eigenvalues of (31), and distinct from identical zero solutions that correspond to the values of parameters $\left(c_1^{(0)}, c_2^{(0)} \right) \in \Lambda_c$ *are called eigenfunctions.*

By direct check we set that for arbitrary values of parameters $(c_1, c_2) \in \Lambda_c$ one of the eigenfunctions is

$$\varphi_0(Q, \mathbf{c}) = \iint_{\bar{\Omega}} F(Q') K(Q, Q', \mathbf{c}) dQ' = f_0(Q, \mathbf{c}). \quad (49)$$

Write, necessary later on, conjugate with (31) equation

$$\psi(Q) = T^*(\mathbf{c})\psi \equiv \frac{F(Q)}{f_0(Q, \mathbf{c})} \iint_{\bar{\Omega}} K(Q, Q', \mathbf{c})\psi(Q')dQ'. \quad (50)$$

At arbitrary $(c_1, c_2) \in \Lambda_c$ one of the eigenfunctions (47) is

$$\psi_0(Q) = F(Q) \quad (51)$$

that unlike $\varphi_1(Q, \mathbf{c})$ does not depend on c_1, c_2.

The existence distinct from identical zero solutions of (31) for arbitrary $(c_1, c_2) \in \Lambda_c$ indicates the existence of

connected components of the spectrum, that coincide with the domain Λ_c. So, the condition of Theorem 2 is not satisfied: $s(\mathsf{A}) \neq \Lambda_c$. To fulfill this condition we eliminate the eigenfunctions (46) from the kernel of (31), namely, we consider the equation

$$\varphi(Q, \mathbf{c}) = \tilde{T}(\mathbf{c})\varphi \equiv \iint_{\overline{\Omega}} \tilde{K}(Q, Q', \mathbf{c})\varphi(Q')\, dQ', \qquad (52)$$

where

$$\tilde{K}(Q, Q', \mathbf{c}) = \frac{F(Q')}{f_0(Q', \mathbf{c})} K(Q, Q', \mathbf{c}) - \frac{\psi_0(Q)\varphi_0(Q', \mathbf{c})}{\|\psi_0\|\,\|\varphi_0\|}. \quad (53)$$

From the Schmidt Lemma [12, p. 132] follows that $\mu = 1$ for any values (c_1, c_2) Equation (53) not be an eigenvalue in the sense of Definition 3, that is $\varphi_0(Q, \mathbf{c})$ will not be an eigenfunction of this equation. The connected component, which coincides with the domain Λ_c and corresponds to function $\varphi_0(Q, \mathbf{c})$, is excluded from the spectrum of operator.

On the basis of (53) we are convinced that the following inequality is valid

$$\iint_{\overline{\Omega}} \iint_{\overline{\Omega}} \left| \tilde{K}(Q, Q', \mathbf{c}) \right|^2 dQ\, dQ' \leq 1 + \alpha_1 \frac{c_1 c_2}{(2\pi)^2} \left[\frac{c_1 c_2}{(2\pi)^2} \mu^2(\overline{G}) + \alpha_F \alpha_{f_0} \right] < +\infty \quad (54)$$

for kernel of operator $\tilde{T}(\mathbf{c})$, where

$$\alpha_1 = \max_{\substack{Q \in \overline{\Omega} \\ \mathbf{c} \in \Lambda_c}} \left| \frac{F(Q)}{f_0(Q, \mathbf{c})} \right|, \qquad \alpha_F = \iint_{\overline{\Omega}} \frac{F(Q)}{\|f_0(Q, \mathbf{c})\|_{L_2}} dQ,$$

$$\alpha_{f_0} = \iint_{\overline{\Omega}} \frac{|f_0(Q', \mathbf{c})|}{\|f_0(Q', \mathbf{c})\|_{L_2}} dQ'.$$

From inequality (54) follows that the operator $\tilde{T}(\mathbf{c})$ is Fredholm with zero index [7]. Moreover, it is completely continuous operator in the space $L_2(\Omega)$.

Functions included in the kernel of (52) admit an analytic continuation in the complex domain Λ_c, if c_1, c_2 is considered as complex parameters. Holomorphy of operator-function $\mathsf{A}(c_1, c_2) = \tilde{T}(c_1, c_2) - I$ follows from the differentiation of the integral operator $\tilde{T}(c_1, c_2)$ on parameters c_1, c_2. In particular, the existence of partial derivatives $\partial\mathsf{A}(c_1, c_2)/\partial c_i$, $i = 1, 2$, at any point $\left(c_1^{(0)}, c_2^{(0)}\right) \in \Lambda_c$, follows from the continuity of the kernel $\tilde{K}(Q, Q', \mathbf{c})$ on the set of variables in the domain $\overline{\Omega} \times \overline{\Omega} \times \overline{\Lambda}_c$ and the existence and continuity of the partial derivatives

$\partial\tilde{K}(Q, Q', c_1, c_2)/\partial c_i$, $i = 1, 2$, in $\overline{\Omega} \times \overline{\Omega} \times \overline{\Lambda}_c$. It is easily to be convinced in this.

Putting in (52) $c_2 = \beta c_1$ (β is a real constant) we consider a nonlinear one-dimensional spectral problem

$$\varphi(Q) = \tilde{\tilde{T}}(c_1)\varphi \equiv \iint_{\overline{\Omega}} \tilde{\tilde{K}}(Q, Q', c_1)\varphi(Q')\, dQ'. \qquad (55)$$

Here $\tilde{\tilde{K}}(Q, Q', c_1) = \tilde{K}(Q, Q', c_1, \beta c_1)$. Since $\tilde{\tilde{T}}(c_1)$ is narrowing of operator $\tilde{T}(c)$, from here follows that $\tilde{\tilde{T}}(c)$ is Fredholm operator with zero index at arbitrary $c_1 \in \Lambda_1$, and the operator-function $\tilde{\mathsf{A}}(\cdot) \equiv (\tilde{\tilde{T}}(\cdot) - I) : \Lambda_1 \to \mathsf{L}(E, E)$ is holomorphic. If $s(\tilde{\mathsf{A}}) \neq \Lambda_1$, from holomorphy of operator-function and Fredholmic of the kernel $\tilde{\tilde{K}}(Q, Q', c_1) = \tilde{K}(Q, Q', c_1, \beta c_1)$ follows satisfaction of conditions of Theorem 2 according to which each point $c_1^{(0)} \in s(\tilde{\mathsf{A}})$ is isolated and is eigenvalue of (55). Respectively, the points $\left(c_1^{(0)}, c_2^{(0)}\right) = \left(c_1^{(0)}, \beta c_1^{(0)}\right)$ are eigenvalues of (52). For the approximate finding the connected components of the spectrum of (52) in the vicinities of the points $\left(c_1^{(0)}, c_2^{(0)}\right)$ we solve the Cauchy problem (46), (47), where as the initial conditions the found solutions of auxiliary problem $\left(c_1^{(0)}, c_2^{(0)}\right) = \left(c_1^{(0)}, \beta c_1^{(0)}\right)$ are used.

Pass to construction of algorithms for numerical finding the eigenvalues of (31) and (52), using the discretization methods [9, 13].

Consider any convergent cubature process [13]:

$$\iint_{\overline{\Omega}} x(Q)\, dQ = \sum_{j=1}^{n} a_{jn} x(Q_{jn}) + \phi_n(x) \quad (n \in \mathbb{N}) \qquad (56)$$

with the coefficients $a_{jn} \in \mathbb{R}$ and nodes $Q_{jn} \in \overline{\Omega}$, $j = 1 \div n$. Reject the remainder term $\phi_n(x)$ in (56) and replace integral in (55) by it. Giving the variable Q the values $Q = Q_{in}$ ($i = 1 \div n$), we have the homogeneous system of linear algebraic equations concerning $u_{1n}, ..., u_{nn} : u_{1n}, ..., u_{nn}$:

$$u_{in} = \tilde{\tilde{T}}_{M_n}(c_1)u_n \equiv \sum_{j=1}^{n} a_{jn} \tilde{K}(Q_{in}, Q_{jn}, c_1) u_{jn} \ (i = 1, n), \quad (57)$$

where $u_{in} = u(Q_{in})$.

Consider the question of solvability of system (57) and convergence of its solutions to solutions of (55).

For this purpose, we consider (55) and the system of equations (57) as operator equations in approaching spaces. The integral equation (55) we shall consider in the Banach space $E = C(\bar{\Omega})$, where $\tilde{\tilde{T}}(c_1) \in \mathsf{L}(E, E)$.

We consider the system of linear equations (57) as opera-

tor equation

$$u_n = \tilde{\tilde{T}}_{M_n}(c_1)u_n \quad (n \in \mathbb{N}) \tag{58}$$

in the Banach space $E_n = C(\bar{\Omega}_n)$, consisting of vectors — grid functions $u_n = (u_{1n}, u_{2n}, ..., u_{nn})$. Here

$$\|u_n\|_{E_n} = \|u_n\|_{C(\bar{\Omega}_n)} = \max_{1 \le i \le n} |u_{in}|$$

and operator $\tilde{\tilde{T}}_{M_n} \in \mathsf{L}(E_n, E_n)$. The connected operators $p_n \in \mathsf{L}(E, E_n)$ are defined by the formula

$$p_n u = (u(s_{1n}), u(s_{2n}), ..., u(s_{nn})) \quad \text{for all } u \in E = C(\bar{\Omega}).$$

Thus, the problem of the eigenvalues $\tilde{A}_\beta(c_1)\mathbf{u} \equiv (\tilde{\tilde{T}}(c_1) - I)\mathbf{u} = 0$ is approximated by the problem of the form $\tilde{A}_{\beta,n}(c_1)\mathbf{u}_n \equiv (\tilde{\tilde{T}}_{M_n}(c_1) - I_n)\mathbf{u}_n = 0$, where $\tilde{A}_{\beta,n}(\cdot): \Lambda_1 \to \mathsf{L}(E_n, E_n)$.

According to [13, p. 99] for operators defined by (55) and (57) the following result is valid: if the kernel $\check{K}: \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ is continuous and the cubature process (56) converges, then $\tilde{\tilde{T}}_{M_n} \to \tilde{\tilde{T}}$ is the compact[1]. Since in these conditions the operators

---

[1] **Comment 1**. Note, for nonlinear one-parameter with respect to $\lambda \in \Lambda \subset \mathbb{C}$ problem of eigenvalues

(i) $\quad x(t) = \int_D K(t, s, \lambda) x(s) ds \quad (A(\lambda) := x - T(\lambda)x = 0)$

and corresponding discrete problem

(ii) $\quad x_{in}(t) = \sum_{j=1}^{n} a_{jn} K(s_{in}, s_{jn}, \lambda) x_{jn}, \quad i = 1, 2, ..., n,$

$(A_n(\lambda)x_n := x_n - T(\lambda)x_n = 0)$

the condition 1-4 of Theorem 2 will be satisfied if the corresponding quadrature process is convergence and the kernel of the integral operator $K(t, s, \lambda)$ satisfies the following conditions:

$K(t, s, \lambda)$ is Holomorphic relative to $\lambda \in \Lambda \subset \mathbb{C}$;

$K(t, s, \lambda)$ and $\partial K(t, s, \lambda)/\partial \lambda$ are continuous relative to $(t, s) \in D \times D$;

$\max_{t \in D} \int_D \left| \frac{K(t, s, \lambda + \Delta\lambda) - K(t, s, \lambda)}{\Delta\lambda} - \frac{\partial K(t, s, \lambda)}{\partial \lambda} \right| ds \to 0$ at $\Delta\lambda \to 0$.

In addition, it is necessary to introduce such condition: there exists such $\lambda \in \Lambda$ which is not eigenvalue of integral equation (i) [13].

$\tilde{\tilde{T}}(c_1) \in \mathsf{L}(C(\bar{\Omega}), C(\bar{\Omega}))$ and $\tilde{\tilde{T}}_{M_n}(\lambda) \in \mathsf{L}(C(\bar{\Omega}_n), C(\bar{\Omega}_n))$ are completely continuous, then the conditions of Theorem 2 are satisfied and on this basis we have:

1) for every $c_1^{(0)} \in s(\tilde{A})$ there exists a sequence $\{c_{1_n}^{(0)}\} \in s(\tilde{A}_n) \ (n > n_0)$ and $c_{1_n}^{(0)} \to c_1^{(0)}$;

2) if $c_{1_n}^{(0)} \to c_1^{(0)} \in \Lambda_1$, $\{c_{1_n}^{(0)}\} \in s(\tilde{A}_n) \ (n \in \mathbb{N}' \subset \mathbb{N})$,

then $c_1^{(0)} \in s(\tilde{A}_\beta)$.

Thus, solving the eigenvalue problem (57) in a finite-dimensional spaces $E_n = C(\bar{\Omega}_n)$, we find approximate eigenvalues that converge to the exact solutions of (55) at $n \to \infty$. Finding the eigenvalues of (57) is reduced, in particular, to finding the roots of the equation

$$\tilde{\Psi}_n(c_1) = \det(\tilde{\tilde{T}}_{M_n}(c_1) - I_n) = 0. \tag{59}$$

Return to two-dimensional spectral problem (52). Applying (56) to (52), we have a system of linear algebraic equations

$$u_{in} = T_{M_n}(c_1, c_2)\mathbf{u} \equiv \sum_{j=1}^{n} a_{jn} \check{K}(Q_{in}, Q_{jn}, c_1, c_2) u_{jn} \ (i = 1 \div n) \tag{60}$$

similar to (57).

Consider finding the solutions of the equation

$$\Psi_n(c_1, c_2) = \det(T_{M_n}(c_1, c_2) - I_n) = 0 \tag{61}$$

as a problem on finding the implicitly given function $c_2 = \gamma(c_1)$, reducing it to Cauchy problem (46) and (47). We use solutions of (59) as the initial conditions (47) since to each isolated root of this equation $c_{1,i}^{(0)}$ corresponds eigenvalue $\mathbf{c}_i^{(0)} = (c_{1,i}^{(0)}, c_{2,i}^{(0)}) = (c_{1,i}^{(0)}, z(c_{1,i}^{(0)})) \in \Lambda_c$ of problem (60). Thus, the initial conditions (50) in the Cauchy problem we determine as $c_{2,i}^{(0)} = \beta c_{1,i}^{(0)}$. If $\frac{\partial \Psi_n}{\partial c_2}(c_{1,i}^{(0)}, c_{2,i}^{(0)}) \ne 0$, we solve the problem (46), (47) in each vicinity of points $(c_{1,i}^{(0)}, c_{2,i}^{(0)}) = (c_{1,i}^{(0)}, \beta c_{1,i}^{(0)}) \in \Lambda_c$. Then we find continuously differentiable function $\gamma_i(c_1)$ that satisfies the condition $\gamma(c_{1,i}^{(0)}) = c_{2,i}^{(0)}$.

In the case when $\mathbf{c}_i^{(0)} = (c_{1,i}^{(0)}, \beta c_{1,i}^{(0)}) \in \Lambda_c$ is a real eigenvalue of the problem (59), $\gamma_i(c_1)$ are a real differentiable functions that describe some smooth

curves in the vicinity of points $\mathbf{c}_i^{(0)} = \left(c_{1,i}^{(0)}, \beta c_{1,i}^{(0)} \right) \in \Lambda_c$. That is, in this case (60) has a line spectrum.

Solving the problems (59), (60), we find a set of parameter values $(c_1, c_2) \in \Lambda_c$ at which from the real initial solution $f_0(Q, \mathbf{c})$ the branching-off of complex solutions of (20) is possible, in which functional (9) takes smaller values than the solution $f_0(Q, \mathbf{c})$.

Show the basic relations for reduction of two-dimensional integral equation of (31) to the corresponding system of linear algebraic equations of the type (60) using the Gauss quadrature formulas that are necessary at calculations by means of the libraries of standard programs.

Put that Gauss formula has m nodes. We introduce into consideration the vector of dimension $n = m^2$

$$\mathbf{u} = \left\{ u_{11}, u_{12}, ..., u_{1,m}, u_{21}, u_{22}, ..., u_{2m}, ..., u_{m1}, u_{m2}, ..., u_{mm} \right\} =$$

$$= \left\{ u(s_1, s_1), u(s_1, s_2), ..., u(s_1, s_m), u(s_2, s_1), u(s_2, s_2), ..., u(s_2, s_m), ... \right.$$

$$\left. ..., u(s_m, s_1), u(s_m, s_2), ..., u(s_m, s_m) \right\} = \left\{ u(s_i, s_j) \right\}_{i,j=1}^n = \left\{ u_{ij} \right\}_{i,j=1}^n$$

$$(62)$$

Reduce four-dimensional matrix to the two-dimensional one with coefficients of the form

$$a_{kl}^{p,q} = b_p \left(\sum_{q=1}^m b_q \frac{F(s_p, s_q)}{f_0(s_p, s_q, c_1, c_2)} K(s_k, s_l, s_p, s_q, c_1, c_2) \right),$$

$$p, q = 1 \div m, \quad k, l = 1 \div m$$

where b_p, b_q are the Gauss nodes.

Note, that solving the Cauchy problem (46), (47) by Runge-Kutta or Adams methods it is necessary to calculate the derivatives of function $\Psi_n(c_1, c_2)$ at points of given grid. Such calculations can be performed by two ways. The first of them consists in direct differentiation of the determinant by (42). The second way is to find approximate derivatives using, in particular, the difference formula [17]:

$$u_{x,i} = \frac{u_{i+1} - u_{i-1}}{2h}.$$

Taking into account the form of right part of (46) to find the derivative at the point $\left(c_1^i, c_2^j \right)$ we obtain the following expression

$$\frac{\frac{\partial \Psi_n \left(c_1^i, c_2^j \right)}{\partial c_1}}{\frac{\partial \Psi_n \left(c_1^i, c_2^j \right)}{\partial c_2}} = \frac{\frac{\Psi_n \left(c_1^{i+1}, c_2^j \right) - \Psi_n \left(c_1^{i-1}, c_2^j \right)}{h}}{\frac{\Psi_n \left(c_1^i, c_2^{j+1} \right) - \Psi_n \left(c_1^i, c_2^{j-1} \right)}{h}} =$$

$$= \frac{\Psi_n \left(c_1^{i+1}, c_2^j \right) - \Psi_n \left(c_1^{i-1}, c_2^j \right)}{\Psi_n \left(c_1^i, c_2^{j+1} \right) - \Psi_n \left(c_1^i, c_2^{j-1} \right)}. \quad (63)$$

Consider the application of this method to numerical finding the connected components of the spectrum (spectral lines) (31) for concrete given functions $F(s_1, s_2)$.

To find the spectral lines of (52) by solving the Cauchy problem (46), (47) it is necessary to find the initial conditions (47), *i. e.* to find the roots of (59). For finding the approximate solutions of this equation using the definition 3, on the basis of (55), we find approximately the set of parameter values c_1 at which $\mu = 1$ is an eigenvalue of equation in the sence of linear eigenvalue problem. Since in the synthesis problems of flat aperture the parameters c_1, c_2 are real and positive, we define the domain Λ_c as:

$$\Lambda_c = \left\{ (c_1, c_2): \ 0 < c_1 < d_1, \ 0 < c_2 < d_2 \right\}.$$

We shall find the approximation to the eigenvalues of (59) on the segments of beams $c_2 = \beta c_1$ belonging to Λ_c as a discrete set of parameter values $\left(c_{1,j}, \beta c_{1,j} \right)$ at which a unit will be an eigenvalue of the matrix (57). For this purpose we select any mesh $c_{1,j}$ on the interval $0 < c_1 \le d_1$. Then, in particular, using Danilevskiy method [18] we find the eigenvalues $\mu_j(c_{1,j})$, those values of parameter $c_{1,j}$ at which $\mu_j(c_{1,j}) \approx 1$ will be approximations to the roots of (59).

The next step consists in finding the corrected roots of (59), considering it as a transcendental equation with respect to c_1. In addition we use the solutions obtained in the preceding paragraph as initial approximation. We use the known methods, in particular, the Muller method [19] to find the roots of (59). As a result, we obtain the corrected values of the eigenvalues of (31) $\left(c_{1,j}^{(0)}, c_{2,j}^{(0)} \right) = \left(c_{1,j}^{(0)}, \beta c_{1,j}^{(0)} \right)$ on the beams $c_2 = \beta c_1$ that belong to Λ_c.

Consider a numerical example of finding the set of eigenvalues $\mu_j\left(c_{1,j}^{(0)} \right)$ by Danilevskiy method. In Fig. 2 the curves corresponding to variuos eigenvalues

$\mu_j(c_1)$ on the beam $c_2 = 0.8c_1$ for $F(s_1, s_2) = 1$ are shown. Vertical lines illustrate the parameter values $c_{1,j}^{(0)}$ for $\mu_j(c_{1,j}^{(0)}) \approx 1$. The approximation to the eigenvalues of the problem (60) is given by bold points in Fig. 3. Approximate and corrected values of the roots of (59) at $\beta = 0.8$ are presented in Table 1.
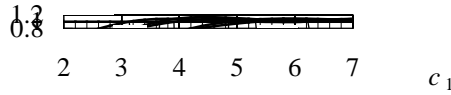

FIG. 2

TABLE 1

$c_I^{(j)}$	Approximate Value	Corrected Value
$c_{.}^{(1)}$	3.14	3.1415927
$c_{.}^{(2)}$	3.65	3.6439078
$c_{.}^{(3)}$	3.92	3.9269908
$c_{.}^{(4)}$	4.88	4.8679838
$c_{.}^{(5)}$	5.35	5.347
$c_{.}^{(6)}$	6.28	6.2831855

Using the corrected values of the roots of (59) as initial data in the Cauchy problem (46), (47) for (60), we find the possible branching line of solutions of (20) which are shown in Fig. 4 for the function $F(s_1, s_2) = 1$.

Note, in the case of rectangular domain $\bar{\Omega}$ the variables can be separated [3] in (31) and $F(s_1, s_2) = F_1(s_1) \cdot F_2(s_2)$. In addition taking into account the form of kernel (15) and (23), we obtain two independent equations
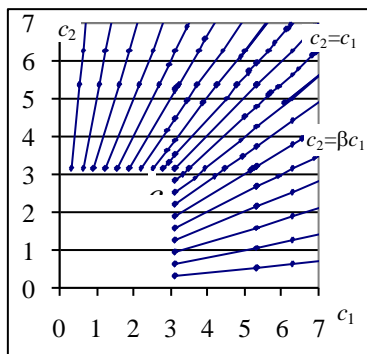
FIG. 3
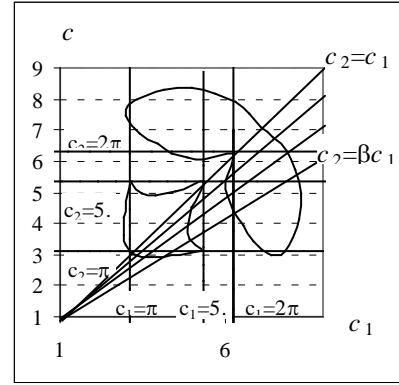
FIG. 4

$$\varphi_j(s_j) = \int_{-1}^{1} \frac{F_j(s_j')}{f_{0,j}(s_j', c_j)} K_j\left(s_j, s_j', c_j\right) \varphi_j(s_j') ds_j' \quad (j = 1, 2).$$

(64)

They describe the possible branching line of solutions of (12), that form a rectangular grid in the domain Λ_c. For function $F(s_1, s_2) \equiv 1$ it is a grid of lines, which is shown in Fig. 4. In the case of nonlinear entering of spectral parameters into the kernel of integral operator (31), obtained by separation of variables of (64), do not describe all solutions of the initial equation (31). This follows, in particular, from analysis of Fig. 3 and Fig. 4. On the last of them we see that besides the direct spectral lines there exist also curves of the spectrum obtained first by solving (31) using the Cauchy problem.

For the case when the variables in the function

$$F(s_1, s_2) = \begin{cases} 2\sqrt{\left(s_1^2 + s_2^2\right)} \cdot \sqrt{\left(1 - \left(s_1^2 + s_2^2\right)\right)}, & \left(s_1^2 + s_2^2\right) \leq 1, \\ 0, & \left(s_1^2 + s_2^2\right) > 1 \end{cases}$$

(65)

are not separated, the spectral lines of (31) are shown in Fig. 5,b. In Fig. 5,a the solutions of auxiliary one-parametrical spectral problem (43) on different beams $c_2 = \beta c_1$ are given. These solutions are the roots of (59) and they are used for solving the Cauchy problem (46), (47) to find the spectral lines presented in Fig. 5,b.

About branching of solutions of synthesis problem of flat rectangular aperture Partial cases

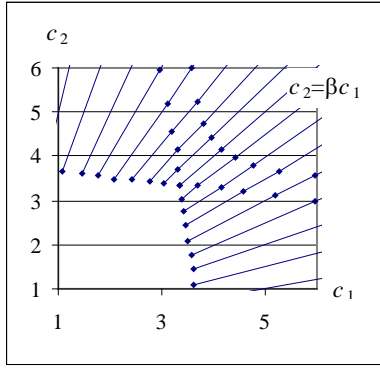Omitting the cumber some intermediate calculations, we shall present the main results obtained in [20, 21]
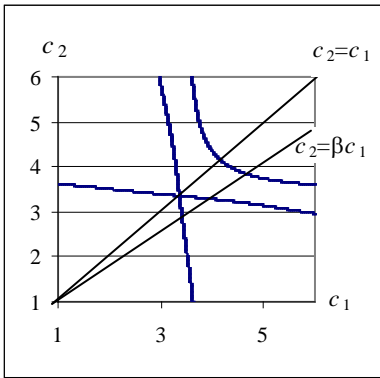
FIG. 5, *A*

FIG. 5, *B*

when we studied branching of solutions of nonlinear two-dimensional integral equations (20) for the case of a flat rectangular aperture. We put also that function $F(s_1, s_2)$ is even on both arguments, the domain $\bar{\Omega}$ is a rectangle and the multiplicity of the eigenvalue $\left(c_1^{(l)}, c_2^{(l)}\right) \in \Lambda_c$ is two on the corresponding spectral lines, where

$$\Lambda_c = \left\{ (c_1, c_2) \in \Lambda_c : 0 < c_1 \le d_1, \ 0 < c_2 \le d_2 \right\}.$$

The investigations of branching of solutions were carried out on the beams $c_2 = \beta c_1$ at $0 < \beta < 1$.

As noted previously, the function

$$\varphi_0\left(s_1, s_2, c_1^{(l)}, c_2^{(l)}\right) = f_0\left(s_1, s_2, c_1^{(l)}, c_2^{(l)}\right)$$

is one of the eigenfunctions of (31) for arbitrary values of the parameters $(c_1, c_2) \in \Lambda_c$.

Except this function of eigenvalues $\mathbf{c}^{(l)} = \left(c_1^{(l)}, c_2^{(l)}\right) = \left(c_1^{(l)}, \beta c_1^{(l)}\right)$ of (31) belonging to the found spectral lines (see Fig. 4 and Fig. 5) there exist eigenfunctions distinct from φ_0. For the case when the multiplicity of an eigenvalue is two, we denote the second eigenfunction as $\varphi_1\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right)$. In

particular, for the first branching point $\left(c_1^{(1)}, 0.8c_1^{(1)}\right)$ (see Fig. 4 and Table 1) at $F(s_1, s_2) \equiv 1$ the second eigenfunction has the form

$$\varphi_1(s_1, s_2, c_1^{(1)}, \beta c_1^{(1)}) = s_2 \cdot f_0(s_1, s_2, c_1^{(1)}, \beta c_1^{(1)}). \tag{66}$$

Giving analogously to (25) to parameter $c_1^{(l)}$ the small disturbance, *i. e.* putting

$$c_1 = c_1^{(l)} + \mu, \quad c_2 = \beta c_1^{(l)} + \beta\mu$$

and taking for (20) similar transformation, we obtain a system of nonlinear integral equations of the Liapunov-Schmidt type concerning small solutions w, ω, similar to (29) and (30):

$$w(Q) = a_{01}(Q, \mathbf{c}^{(l)})\mu +$$

$$+ \sum_{m+n+p \ge 2} \mu^p \iint_{\bar{\Omega}} A_{mnp}(Q, Q', \mathbf{c}^{(l)}) w^m(Q') \omega^n(Q') dQ', \tag{67}$$

$$\omega(Q) - \iint_{\bar{\Omega}} F(Q') K(Q, Q', \mathbf{c}^{(l)}) \frac{\omega(Q')}{f_0(Q', \mathbf{c}^{(0)})} dQ' =$$

$$= \sum_{m+n+p \ge 2} \mu^p \iint_{\bar{\Omega}} B_{mnp}(Q, Q', \mathbf{c}^{(l)}) w^m(Q') \omega^n(Q') dQ', \tag{68}$$

where

$$a_{01}(Q, \mathbf{c}^{(0)}) = \iint_{\bar{\Omega}} A_{001}(Q, Q', \mathbf{c}^{(l)}) dQ'.$$

The left part contains two-dimensional linear integral equation of the type (31). Taking on the system transformations on the basis of general branching theory of solutions [12], we obtain that two complex conjugate solutions branch-off from initial (real) solution (23) at the point $\left(c_1^{(l)}, \beta c_1^{(l)}\right)$. In the first approximation they have the form

$$f_{1,2}^{(1)}(s_1, s_2, c_1, \beta c_1) = f_0\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right) +$$

$$+ \left(a\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right) + \alpha_{020}^{(1)}\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right) h_1^2\right) \mu \pm$$

$$\pm i \frac{\varphi_1\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right)}{\left\|\varphi_1\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right)\right\|} h_1 \mu^{1/2} + O(\mu^{3/2}). \tag{69}$$

Indicate some characteristics of branching-off solutions of (12) for the case of symmetric aperture $\bar{G}$ and even on both arguments (or on one argument) of the function $F(s_1, s_2)$. In the first approximation the properties of solutions follow from (66). They are defined by the feature of functions

$$f_0\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right), \quad a\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right),$$

$\alpha_{020}^{(1)}\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right)$ and the property of the eigenfunction $\varphi_1\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right)$ which corresponds to the l-th point of branching of solution $\left(c_1^{(l)}, \beta c_1^{(l)}\right)$. The last two solutions have the same type of parity as the function $f_0\left(s_1, s_2, c_1^{(l)}, \beta c_1^{(l)}\right)$. It is confirmed also by the general properties $1^{\circ} - 3^{\circ}$ of the solutions of (12) given previously in II.

Numerical Algorithms of Finding the Optimal Solutions of Base Equations of Synthesis

Give one of the iterative processes for numerical finding the solutions of (20), which is based on the method of successive approximations [3]:

$$u_{n+1}(Q) = B_1(u_n, v_n) \equiv \iint_G K(Q, Q', c) F(Q') \frac{u_n(Q')}{\sqrt{u_n^2(Q') + v_n^2(Q')}} dQ'$$
,
$$v_{n+1}(Q) = B_2(u_n, v_n) \equiv \iint_G K(Q, Q', c) F(Q') \frac{v_n(Q')}{\sqrt{u_n^2(Q') + v_n^2(Q')}} dQ'$$

$$(n = 0, 1, \dots). \qquad (70)$$

We substitute in (10) functions $\arg f_n(s_1, s_2) =$ $= \operatorname{arctg}\left(v_n(s_1, s_2)/u_n(s_1, s_2)\right)$ obtained through successive approximations (70). Thus we obtain a sequence of functions

$$U_n(x, y) = \frac{c_1 c_2}{(2\pi)^2} \iint_{\bar{\Omega}} F(s_1, s_2) e^{i\left[\arg f_n(s_1, s_2) - (c_1 x s_1 + c_2 y s_2)\right]} ds_1 ds_2 \quad (71)$$

which is denoted as $\{U_n\}$. For a sequence $\{U_n\}$ the theorem 4.2.1 with [3] is valid. It indicates that the sequence $\{U_n\}$ is a relaxational for (7), and numerical sequence $\{\sigma(U_n)\}$ is convergent.

In the case of even on both (or on one) arguments function $F(s_1, s_2)$ and symmetric domains $\bar{G}$ and $\bar{\Omega}$ it is expedient to use the property of invariance of integral operators $B_1(u, v)$ and $B_2(u, v)$ in (20) relatively the type of parity functions $u(s_1, s_2)$, $v(s_1, s_2)$ for the iteration process (41). Functions u, v having some type of parity on the corresponding argument belong to invariant sets Φ_{ij}, $\Phi_{k\ell}$ of space $\mathbf{C}(\Omega)$. Here indices i, j, k, ℓ take the values 0 or 1. In particular, if $u(s_1, s_2) \in \Phi_{01}$ then $u(-s_1, s_2) = u(s_1, s_2)$, and $u(s_1, -s_2) = -u(s_1, s_2)$. By direct check we make sure that there are such inclusions:

$$B_1(\Phi_{ij} \bigcup \Phi_{k\ell}) \subset \Phi_{ij}, \ B_2(\Phi_{ij} \bigcup \Phi_{k\ell}) \subset \Phi_{k\ell},$$

$$\mathbf{B}(\Phi_{ij} \bigcup \Phi_{k\ell}) \subset \Phi_{ij} \bigcup \Phi_{k\ell}.$$

From these relations follows the possibility of existence of fixed points of the operator $\mathbf{B}$ belonging to the corresponding invariant set, that is to the solutions of (20) and, respectively, to (10) and (12).

Analysis of Numerical Results

First we consider the analysis of branching of solutions of (12) with the kernel (15) for the given DP $F(s_1, s_2) \equiv 1$ on the beam $c_2 = 0.8 c_1$. The lines of possible branching of solutions were presented in Fig. 4. The first few points of branching of solutions on the ray $c_2 = 0.8 c_1$ are given in Table 1. From Fig. 4 and Table 1 follows that the point $\left(c_1^{(1)}, \beta c_1^{(1)}\right) = (\pi, 0.8\pi)$ is the first branching point of the initial solution $f_0(s_1, s_2)$. The eigenfunction of linear homogeneous integral equation (31)

$\tilde{\varphi}_1\left(s_1, s_2, \pi, 0.8\pi\right) = s_2 \cdot f_0(s_1, s_2, \pi, 0.8\pi)$ different from φ_0 and corresponding to that point is even on the argument s_1 and odd on the argument s_2. Accordingly branched-off at point $\left(c_1^{(0)}, \beta c_1^{(0)}\right) = (\pi, 0.8\pi)$ two complex conjugate solutions have the following properties: $\operatorname{Re} f_{1,2}^{(1)}\left(s_1, s_2, c_1^{(0)}, \beta c_1^{(0)}\right)$ is even on both arguments, and $\operatorname{Im} f_{1,2}^{(1)}\left(s_1, s_2, c_1^{(0)}, \beta c_1^{(0)}\right)$ is odd function on argument s_2. Respectively the functions $\arg f_{1,2}^{(1)}\left(s_1, s_2, c_1^{(0)}, \beta c_1^{(0)}\right)$ are odd concerning s_2. In particular, it is confirmed by the results of numerical experiment.

The distribution function of oscillating velocity $U(x, y)$ on the surface of the antenna which is determined by formula (71) depends on the properties of functions $F(s_1, s_2)$ and $\arg f(s_1, s_2)$. In the case of symmetric domains $\bar{G}$ and $\bar{\Omega}$, even $F(s_1, s_2)$ and function $\arg f(s_1, s_2)$ odd on one of arguments, $U(x, y)$ is a real and nonsymmetric function in respect to one of the coordinate axes of aperture $\bar{G}$.

Fig. 6 presents the value of the functional σ_F, which it takes on the primary (curve **1**) and branching-off solutions (curves **2, 3, 4**). Curve **2** corresponds to branching-off solutions with phase DP odd on s_2. Optimal distribution function of oscillating velocity in

aperture corresponding to this type of solutions is real and asymmetric relatively the axis OX. However, formed by it synthesized amplitude DP is symmetric.
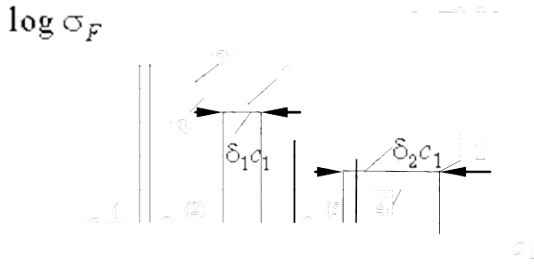

FIG. 6 THE VALUES OF FUNCTIONAL ON VARIOUS TYPES OF SOLUTIONS

Curve **3** corresponds to branching-off solutions at the point $\left(c_1^{(2)}, \beta c_1^{(2)}\right) = (3.6439, 2.91512)$ with the following properties:

$$\begin{cases} \arg f_{1,2}^{(2)}\left(-s_1, s_2\right) = -\arg f_{1,2}^{(2)}\left(s_1, s_2\right), \\ \arg f_{1,2}^{(2)}\left(s_1, -s_2\right) = -\arg f_{1,2}^{(2)}\left(s_1, s_2\right). \end{cases} \quad (72)$$

Eigenfunctions of (31) corresponding to this branching point have the form:

$$\varphi_0\left(s_1, s_2\right) = f_0(s_1, s_2, 3.6439, 2.91512),$$

$$\varphi_1\left(s_1, s_2\right) = s_1 s_2 f_0(s_1, s_2, 3.6439, 2.91512).$$

Curve **4** corresponds to the branching-off complex conjugate solutions at the point $\left(c_1^{(5)}, 0.8c_1^{(5)}\right) = (5.347, 4.2776)$ with the following properties:

$$\begin{cases} \arg f_{1,2}^{(5)}\left(-s_1, s_2\right) = \arg f_{1,2}^{(5)}\left(s_1, s_2\right), \\ \arg f_{1,2}^{(5)}\left(s_1, -s_2\right) = \arg f_{1,2}^{(5)}\left(s_1, s_2\right), \end{cases} \quad (73)$$

that is, the optimal phase DP is even on both arguments.

From the analysis of Fig. 6 we see that solution with the properties (72) is the most effective on the interval $\left(c_1^{(2)}, c_1^{(5)}\right)$ and solution with properties (73) − on the interval $\left(c_1^{(5)}, 8.0\right)$ As an example, further, the results of synthesis of given amplitude DP $F(s_1, s_2) \equiv 1$ at $c_1 = 7.0$ on the beam $c_2 = 0.8c_1$ corresponding to different types of branching-off solutions are presented.

The solution with the properties (73) is the most effective at $c_1 = 7.0$. The synthesized amplitude DP $\left|f\left(s_1, s_2\right)\right|$ and phase DP $\arg f\left(s_1, s_2\right)$ odd on s_1, are given in Fig. 7,a and Fig. 7,b, respectively. Function of distribution of oscillating velocity in aperture (Fig. 8), which forms this DP, is real, but asymmetric concerning the axis OX.

In Fig. 9 and Fig. 10 the results of the synthesis of a given amplitude DP $F(s_1, s_2) \equiv 1$ at $c_1 = 7.0$, corresponding to the phase DP with properties (72), are presented.

The most effective solutions of the synthesis of a given amplitude DP $F(s_1, s_2) \equiv 1$ at $c_1 = 7.0$, corresponding to phase DP with properties (73), are shown in Fig. 11 and Fig. 12.
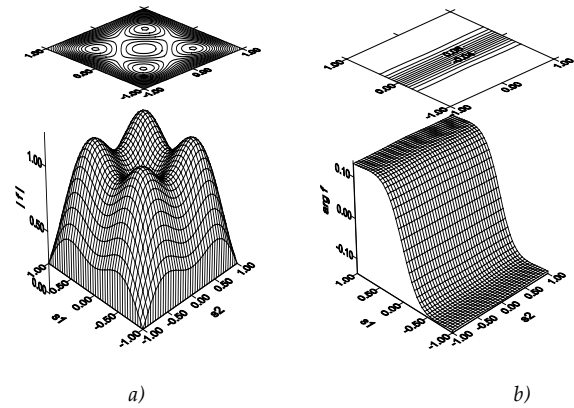

a)   *b)*

FIG. 7 SYNTHESIZED AMPLITUDE *(A)* AND ODD PHASE *(B)* DP'S

In Fig. 13 we present the results of synthesis of the given four-petalous DP $F(s_1, s_2) = \left|\sin\left(\pi s_1\right)\right| \cdot \left|\sin\left(\pi s_2\right)\right|$ at $c_1 = c_2 = 7.0$, which is responsible for different types of solutions. The first type of solution has odd phase DP (Fig. 14,a) and asymmetric relatively to axis OY but real amplitude distribution of the oscillating velocity in aperture. Synthesized DP corresponding to this solution is shown in Fig. 13,b.
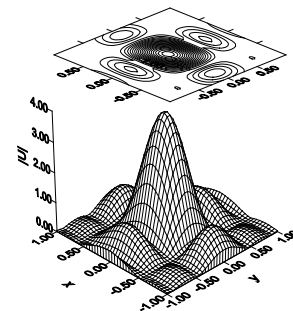

FIG. 8 THE OPTIMAL DISTRIBUTION OF OSCILLATING VELOCITY, THAT FORMS, AMPLITUDE AND PHASE DP'S PRESENTED IN FIG. 7
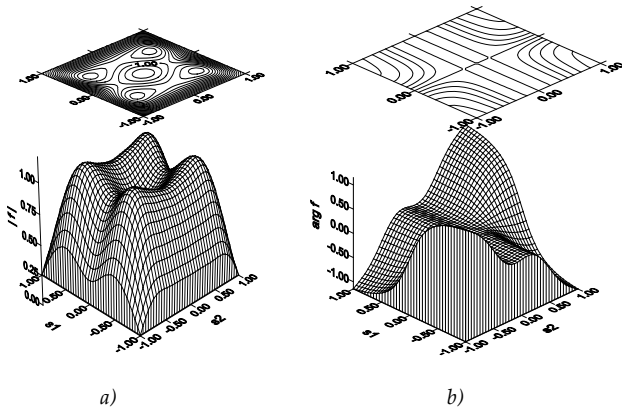
a)   *b)*

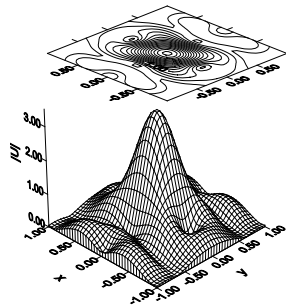FIG. 9 THE SYNTHESIZED AMPLITUDE *(A)* AND ODD ON BOTH ARGUMENTS PHASE *(B)* DP'S

FIG. 10 THE OPTIMAL DISTRIBUTION OF OSCILLATING VELOCITY, THAT FORMS, AMPLITUDE AND PHASE DP'S PRESENTED IN FIG. 9
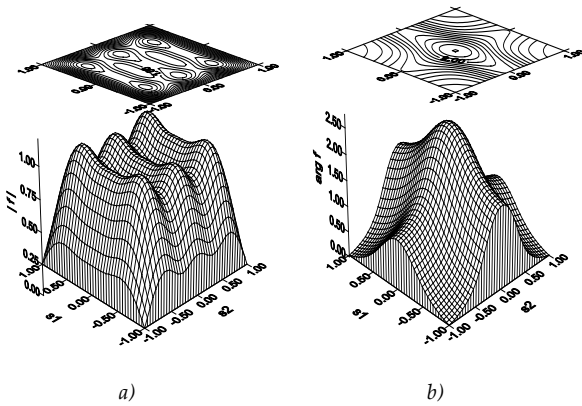

a)   *b)*

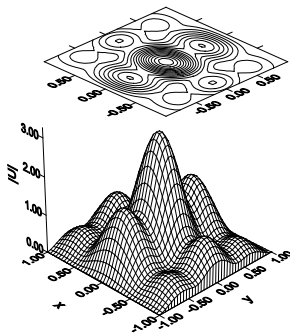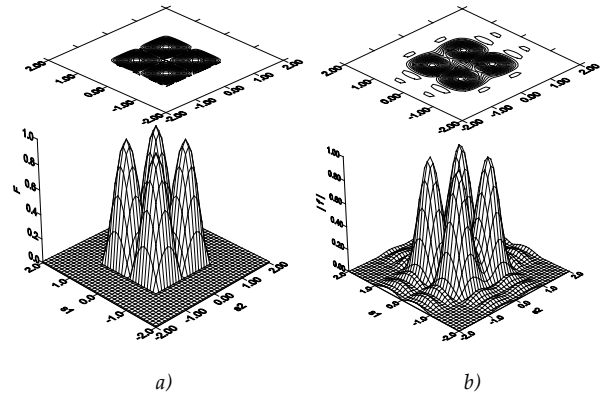FIG. 11 SYNTHESIZED AMPLITUDE *(A)* AND EVEN ON BOTH ARGUMENTS PHASE *(B)* DP'S

FIG. 12 THE OPTIMAL DISTRIBUTION OF OSCILLATING VELOCITY, THAT FORMS, AMPLITUDE AND PHASE DP'S PRESENTED IN FIG. 11

a)   *b)*

FIG. 13 THE PRESCRIBED *(A)* AND SYNTHESIZED *(B)* AMPLITUDE DP'S

a)   *b)*
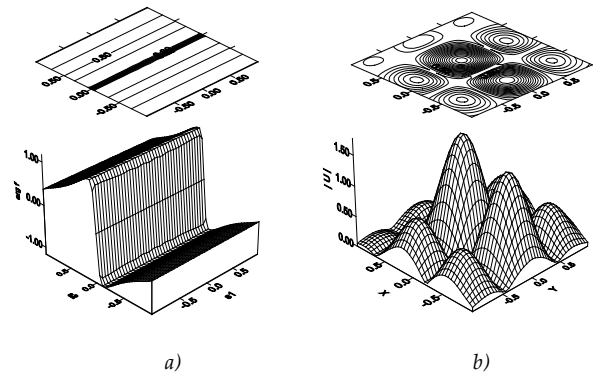
FIG. 14 SYNTHESIZED PHASE DP *(A)* AND THE OPTIMAL DISTRIBUTION OF OSCILLATING VELOCITY IN APERTURE *(B)*

The second type of solution has phase DP even on both arguments (Fig. 15,*b*) and amplitude distribution of the oscillating velocity in aperture symmetric relatively two axes (Fig. 15,*a*).

Comparing the synthesized amplitude DP's (Fig. 13,*b* and Fig. 15,*a*) that coincide with each other to with in graphic precision and correspond to different types of solutions, we make sure that one and the same (or close to each other the amplitude DP) can be formed by different distributions of oscillating velocity in the aperture of antenna.

Numerical examples of synthesis of prescribed amplitude funnel-shaped DP defined by formula (65) in domain $\bar{\Omega}$ are shown on Fig. 17 and Fig. 18. The lines of possible branching of solutions of (31) are given in Fig. 5. The prescribed DP and optimal synthesized DP are shown in Fig. 17,*a* and Fig. 17,*b*, respectively, at $c_1 = 9.25$, $c_2 = 7.4$. The optimal function of distribution of oscillating velocity $U(x, y)$ corresponding to synthesized DP shown in Fig. 17,*b* is presented in Fig. 18.
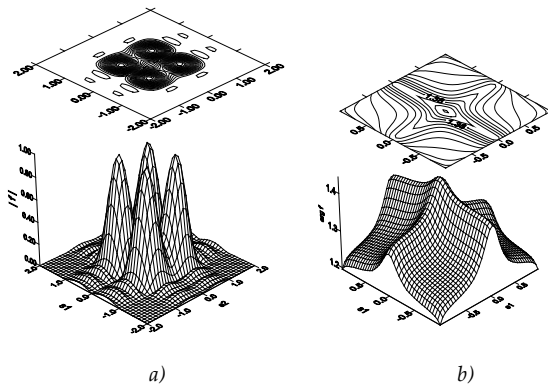
a)   *b)*

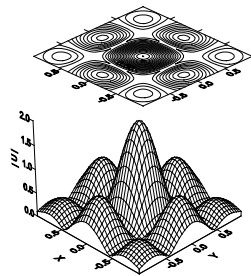FIG. 15 SYNTHESIZED AMPLITUDE *(A)* AND PHASE *(B)* DP'S

FIG. 16 THE OPTIMAL DISTRIBUTION OF OSCILLATING VELOCITY THAT FORMS, AMPLITUDE AND PHASE DP'S PRESENTED IN FIG. 15
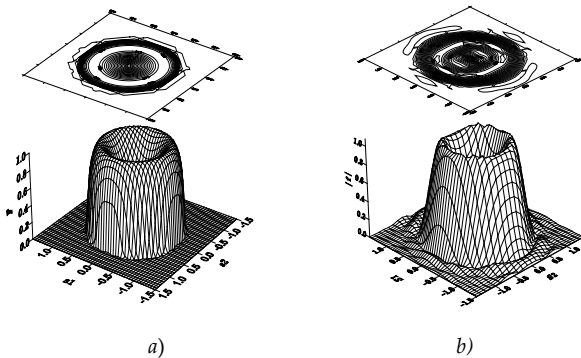

a)   *b)*

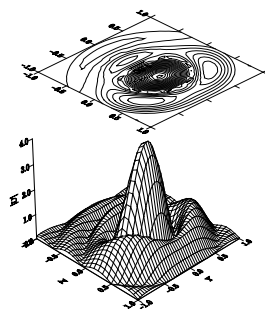FIG. 17 THE PRESCRIBED *(A)* AND SYNTHESIZED *(B)* AMPLITUDE DP'S

FIG. 18 THE OPTIMAL DISTRIBUTION OF OSCILLATING VELOCITY THAT FORMS, AMPLITUDE AND PHASE DP'S PRESENTED IN FIG. 17

Conclusions

- Note that the presence of nonuniqueness of solutions in nonlinear synthesis problems of antennas according to the prescribed amplitude DP greatly complicates the mathematical investigations of optimal solutions. It provides an opportunity in practice to choose the most effective solution that has a relatively simple physical realization.

- The results of synthesis confirming the effectiveness of branching-off solutions are shown in Fig. 6. Here on the edge of segments $\delta_1 c_1$ and $\delta_2 c_1$ functional takes the same values on the primary (real) and branching-off (complex) solutions. From here follows that efficiency of the synthesis on the branching-off solution in comparison with the efficiency of synthesis on the real solution can be achieved at smaller values of the parameter c_1 characterizing the linear dimension of aperture. Thus, the linear dimension of the antenna can be decreased on the value $\delta_1 c_1$ or $\delta_2 c_1$ at realization of branching-off solution. In addition the area of antenna is reduced accordingly. It significantly influences on technical characteristics of antenna and the expenses of its realization.

- In the case of given amplitude DP $F(s_1, s_2)$ even on both arguments the investigations of three-dimensional type of branching of solutions of (20), (for example, the point $\left(c_1^{(5)}, \beta c_1^{(5)} \right)$, see Table 1), is advisable to carry out separately in classes of even and odd functions (where there is two-dimensional type of branching) using the invariance property of nonlinear integral operators $\mathbf{B} = \left(B_1, B_2 \right)$ in (20) concerning the type of parity of phase DP.

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